components is obtained. Except for the last downstream station, excellent agreement between the two methods is also found for the transverse components with the accuracy comparable to the axial data. Agreement with the theory is not as good in the transverse direction, but Higuchi and Rubesin[3] attribute this to limitations in the turbulence model.

At the last downstream station, poorer agreement in the transverse direction between the two methods is observed, and the uncertainty is larger for the laser interferometer data. This can be attributed to three problems that arise with the oil-flow method when measuring transverse skin-friction components nearly perpendicular to the flow direction.

First, since the skin-friction component becomes small, the gravity correction becomes large, and the approximations in the oil-flow theory become less accurate. For example, the gravity correction was 13% at the last station.

Second, small errors in the applied oil-line angle result in large errors in the measured skin friction. For example, when ϕ is 90 deg (a transverse measurement), the error in skin friction is $\delta\phi/\tan\gamma$, which approaches infinity as γ approaches 0 deg for any finite error, $\delta\phi$, in oil-application angle.

Third, the oil-flow path length from the oil leading edge to the downstream beam measurement point becomes large, causing persistent oil surface waves. An attempt was made to reduce the gravity and surface-wave effects at the last station by making measurements with 50-cS oil rather than the normally used 200-cS oil. Although the less viscous 50-cS oil thinned faster and reduced the effects described above, fewer visible fringes were available for the data reduction and thus accuracy was not improved. Consequently, the 3 deg flow angle at this position is probably close to the lower limit for measuring transverse skin friction accurately in three-dimensional flows.

The measured skin-friction components in Fig. 2 may be used to compute the local surface-flow angle γ using $\tan\gamma = C_{fz}/C_{fx}$. The results are compared in Fig. 3 with a theoretical computation and angles measured from surface oil-flow patterns by Higuchi and Rubesin.[3] As expected from the good agreement in Fig. 2, the two skin-friction methods also agree in determining flow angle. Furthermore, they agree with the oil-flow pattern data over the entire range. Note the accuracy of the laser interferometer method even at the last downstream station where there is some error in transverse skin friction. These comparisons lend additional confidence in the skin-friction data previously presented for both methods. On the other hand, the theoretical curve for flow angle in Fig. 3 lies above the measurements over the region tested. This is expected from the previous disagreement found for the transverse skin friction in Fig.2.

Conclusions

The preceding results establish the accuracy and utility of the laser interferometer skin-friction method in three-dimensional flows with unknown direction, at least for flows such as the present one with spanwise similarity. Limitations to the method were found for transverse skin-friction measurements close to the perpendicular to the flow direction, but components outside of 3 deg to the flow normal were measured successfully. In addition, the method is limited to wind tunnels with steady run times of approximately 30 s or longer, in order to accurately measure the flow rate of the oil placed on a test surface.

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Design Sensitivity Analysis Strain Energy via Distribution

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Nomenclature

b, b_m = vector of design variables and the mth component
$D_{\alpha\beta}$ = compliance tensor
g, g_j = general form of constraints and its component
G, G_j = $g = G(b,u)$ [see Eq. (2)]
n_j = unit vector perpendicular to surface S
N_b = number of design variables
N_g = number of constraints
N_ℓ = number of loading cases necessary to compute displacements
N_p = number of specified forces
N_r = total number of redundancies
N_u = number of displacements in constraint function G
P, P_i = effective load vector and the ith component
ΔP_i = increment of P_i
q^r = reactant component of Q_α
Q_α = generalized stress
Q_α^w = free component of Q_α
S = total surface of a body or structure $= (S_u + S_p)$
S_p = boundary surface where forces are prescribed
S_u = boundary surface where displacements are prescribed
u, u_i = nodal displacement vector and the ith component
U = total strain energy stored in a loaded system
U_m = strain energy stored in volume V_m
V = total volume of an elastic body or structure
V_m = subset of the total volume V
x_j = generic point of the reference configuration
γ = ratio of the computing cost
λ_α^r = linear function of x_j corresponding to q^r

Introduction

IN the present Note, we set out to develop an alternative method for the determination of design sensitivity coefficients of elastic structures based on Castigliano's theorem. The method requires only information on strain energy distribution; hence, it offers an advantage of ease of implementation with an existing finite element program with no modification to the source code.

Three different approaches of computing sensitivity coefficients have been proposed and studied extensively by others.[1-4] They are[5]: 1) the virtual load method (or dummy load method); 2) the state space method; and 3) the design space method.

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Let us consider a general structural problem in terms of displacements. The equilibrium condition for a finite element method of a structure is given as:

$$K(b)u=P \tag{1}$$

where P is assumed to be independent of design variables. A general form of constraints can be written as functions of both design variables b, and nodal displacements u; that is,

$$g=G(b,u)\leq 0 \tag{2}$$

where the constraint vector G is generally a nonlinear function of b and u, and can be used to represent stresses, displacements, or other constraints that depend on b and u. Taking the total derivative of g_j with respect to b_m yields

$$\frac{\partial g_j}{\partial b_m}=\frac{\partial G_j}{\partial b_m}+\frac{\partial G_j}{\partial u_i}\frac{\partial u_i}{\partial b_m}$$

$$i=1,\ldots,N_u;\quad j=1,\ldots,N_g;\quad m=1,\ldots,N_b \tag{3}$$

The essential part of design sensitivity analysis is computation of the term $\partial u_i/\partial b_m$, either directly or indirectly. The basic differences among the three methods lie in the procedure for obtaining this term. It should be noted that the final expression of Eq. (3) is identical for all three methods, and can be written as[6,7]

$$\frac{\partial g_j}{\partial b_m}=\frac{\partial G_j}{\partial b_m}-\frac{\partial G_j}{\partial u_i}K^{-1}\left(\frac{\partial K}{\partial b_m}u_i\right) \tag{4}$$

Basic Concepts of Present Approach

Theoretical Background

Consider an elastic body or structure fixed at boundary points over surface S_u and subjected to specified forces $P_i\,(i=1,\ldots,N_p)$ over the remaining portion S_p (Fig. 1). No body forces are considered. If the material follows Hooke's law and the deformations are small, the strain energy U stored in the loaded system will be equal to the work done by the applied forces and independent of the order in which they are applied. If the displacement u_i is required at a point on a structure at which a load P_i acts, then Castigliano's theorem states that

$$u_i=\frac{\partial U}{\partial P_i} \tag{5}$$

where the displacement u_i is measured in the direction of P_i. The strain energy U may be expressed in terms of generalized stresses as

$$U=\frac{1}{2}\int_V D_{\alpha\beta}Q_\alpha Q_\beta \,\mathrm{d}V \tag{6}$$

Let x_j denote a point in the reference configuration, which can be either a point on the centerline of a one-dimensional structure ($j=1$), a point on the middle surface of a plate or shell ($j=1,2$), or a general point in a three-dimensional body ($j=1,2,3$). Since a structure is, in general, statically indeterminate, one may divide the generalized stress $Q_\alpha(x_j)$ at any point x_j into two parts:

$$Q_\alpha(x_j)=Q_\alpha^w(x_j)+\sum_{r=1}^{N_r}\lambda_\alpha^r(x_j)q^r \tag{7}$$

where $Q_\alpha^w(x_j)$ and q^r are free and reactant components, respectively. $\lambda_\alpha^r(x_j)$ $(r=1,\ldots,N_r)$ are linear functions of x_j, and N_r is the total number of redundancies. Then, substituting Eq. (7) into Eq. (6) and using the principle of virtual work, one can prove that

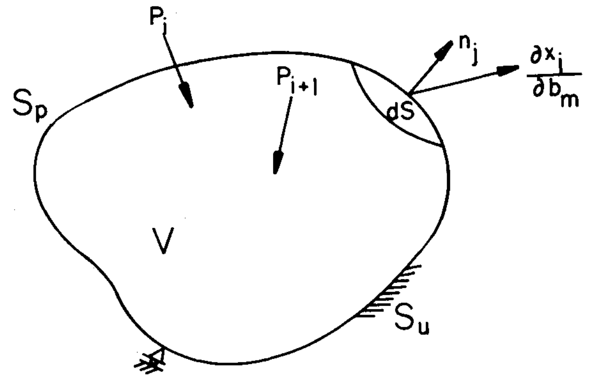

Fig. 1 Reference volume in space.

$$\frac{\partial U}{\partial q^r}\equiv\int_V D_{\alpha\beta}\lambda_\alpha^r Q_\beta \,\mathrm{d}V=0 \tag{8}$$

This is Castigliano's Theorem of Compatibility,[8] often referred to as the principle of minimum strain energy. Equation (8) implies that the quantity U is minimum with respect to the values of each of the redundancies; Eq. (8) thus yields exactly N_r equations from which the values of the redundancies may be found.

Design Sensitivity Analysis

The objective of this section is to derive a relationship between the sensitivity coefficient $\partial u_i/\partial b_m$ and the strain energy U, where b_m $(m=1,\ldots,N_b)$ is the mth design variable. Using Eqs. (5) and (6), we may express $\partial u_i/\partial b_m$ as

$$\frac{\partial u_i}{\partial b_m}=\frac{\partial}{\partial P_i}\left[\frac{1}{2}\int_S D_{\alpha\beta}Q_\alpha Q_\beta\frac{\partial x_j}{\partial b_m}n_j\mathrm{d}S\right.$$

$$\left.+\frac{1}{2}\int_V\frac{\partial D_{\alpha\beta}}{\partial b_m}Q_\alpha Q_\beta\mathrm{d}V+\int_V D_{\alpha\beta}\frac{\partial Q_\alpha}{\partial b_m}Q_\beta\mathrm{d}V\right] \tag{9}$$

where n_j is the normal vector to the boundary surface S as shown in Fig. 1. Here Eq. (9) may be considered as material derivative of volume integral.[9]

Equation (9) can be greatly simplified if one chooses certain types of design variables. For example, choice of cross-sectional properties (e.g., material property, area, moments of inertia, etc.) of either one-dimensional beam or two-dimensional plate/shell structures makes the first term of Eq. (9) identical to zero.

And since the free components Q_α^w in Eq. (7) are the solutions of the statically determinate structures, they are independent of cross-sectional properties, which gives us [Eq. (7)]

$$\frac{\partial Q_\alpha}{\partial b_m}=\sum_{r=1}^{N_r}\lambda_\alpha^r\frac{\partial q^r}{\partial b_m} \tag{10}$$

Then, using the minimum strain energy principle [Eq. (8)] and Eq. (10), it can be shown that the last term of Eq. (9) also vanishes. After all, one can rewrite Eq. (9) as

$$\frac{\partial u_i}{\partial b_m}=\frac{\partial}{\partial P_i}\left[\frac{1}{2}\int_{V_m}\frac{\partial D_{\alpha\beta}}{\partial b_m}Q_\alpha Q_\beta\mathrm{d}V\right] \tag{11}$$

It should be noted that integration in Eq. (11) need only be performed over the region V_m in which $D_{\alpha\beta}$ depends on b_m. Moreover, if one can express the compliance tensor $D_{\alpha\beta}$ as inversely proportional to b_m ($D_{\alpha\beta} \propto 1/b_m$) in the region V_m, then Eq. (11) can be further simplified, such as

$$\frac{\partial u_i}{\partial b_m} = -\frac{1}{b_m}\frac{\partial U_m}{\partial P_i} \tag{12}$$

where U_m is the strain energy stored in the volume V_m due to the external loads P_i ($i = 1,...,N_p$). If a finite element technique is used, the volume V_m is divided, in general, into a finite number of elements. In this case, U_m is the sum of all the element strain energies. Substituting Eq. (12) into Eq. (3) gives us

$$\frac{\partial g_j}{\partial b_m} = \frac{\partial G_j}{\partial b_m} - \frac{\partial G_j}{\partial u_i}\left(\frac{1}{b_m}\frac{\partial U_m}{\partial P_i}\right) \tag{13}$$

Note that Eq. (13) is of a form similar to Eq. (4).

By comparing Eq. (13) with Eq. (4), one can realize that the current method requires element strain energy terms in order to compute sensitivity coefficients, whereas the other methods require element stiffness matrices. Since the strain energy terms are commonly produced by finite element programs, the current method offers an advantage of easy implementation *without* modification to the source code.

Discussion and Conclusions

Before proceeding, it is useful to propose a procedure for implementing the present approach in conjunction with existing finite element programs (e.g., MSC/NASTRAN, etc.):

1) Choose a displacement component u_i of which one wishes to obtain derivatives with respect to b_m ($m = 1,...,N_b$).

2) Apply two sets of loadings ($P_1...P_i...P_{N_p}$) and ($P_1,...,P_i + \Delta P_i,...,P_{N_p}$) to a structural model. It should be noted that other boundary conditions are identical except for the loading condition at point i. After completing the analysis, read strain energies for both loading cases [e.g., $U_e(P_i)$ and $U_e(P_i + \Delta P_i)$] from the computer output. Here $e = 1,...,N_e$, where N_e denotes the total number of elements of a structural model.

3) Compute

$$\frac{\partial U_e}{\partial P_i} = \frac{U_e(P_i + \Delta P_i) - U_e(P_i)}{\Delta P_i} \tag{14}$$

for all elements, which depend on any of b_m.

4) Calculate

$$\frac{\partial}{\partial b_m}\left(\frac{\partial U_m}{\partial P_i}\right) = -\frac{1}{b_m}\sum_{e=1}^{N_m}\left(\frac{\partial U_e}{\partial P_i}\right) \tag{15}$$

where N_m ($< N_e$) is the number of elements that contain the mth design variable [see Eq. (13)].

5) Repeat steps 1-4 with a different displacement, u_{i+1}.

Step 2 suggests that two sets of loadinz cases are needed for each displacement u_i. Therefore, if there are N_u displacements in the constraint functions [Eq. (3)], one must solve $2N_u$ sets of loadings, which means, theoretically, $2N_u$ separate finite element analyses. The actual cost, however, is much less than $2N_u$ times the cost of a single finite element analysis, because the decomposed stiffness matrix K can usually be used for multiple loading cases (N_ℓ). Note that the cost of the present approach can be shown to be independent of the number of design variables N_b.

Thus far, an alternative method for computing design sensitivity coefficients based on Castigliano's theorem has been proposed. Although this theorem has been known for many years, its application to design sensitivity analysis is new. Since the cost of the present method is independent of the number of design variables, the efficiency appears to compare favorably with other methods. In addition, it offers advantages of both simple integration using existing finite element programs and a potential for extension to nonlinear problems.

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Methods of Reference Basis for Identification of Linear Dynamic Structures

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Introduction

THE initial approach to the identification of structures was to use test data only. For structures with a small number of degrees-of-freedom this approach can be satisfactory. However, for complex structures such as aircraft vehicles, test data are incomplete by necessity. In Refs. 1-6 it was proposed to add to the vibration test data independently obtained analytical mass, $A(n \times n)$, and stiffness, $K(n \times n)$, matrices. Now the available data are redundant and usually will not comply with the theoretical requirements of a physically possible structure. Hence, some of the data must be corrected. Some of the data can be taken as a reference basis and used to correct the remaining data.

The rigid body modes, $R(n \times r)$, are theoretically well defined and must be kept unchanged. The parts of the analytically obtained mass matrix A connected with the rigid body modes can be obtained in an independent manner and must also be kept unchanged.[5] It is widely accepted that the measured frequencies constitute the most accurate test data. Hence, the measured frequencies, represented by $\Omega^2(m \times m)$,

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